Fortification Engineering
as a Context for the Sine, Cosine, and Tangent Rules[1]

If you have the opportunity to visit the Netherlands, we highly recommend visiting the old fortified towns of Heusden, Naarden or Bourtange. In such places, mathematics is literally there for the taking. The fortress of Bourtange is a successful reconstruction. In figure 1, the old road runs from top to bottom, along the moat, through the fortress. It was once one of three roads from the city of Groningen to Germany. An army first had to conquer this fortress before it could invade Groningen. In this article, Henk Hietbrink explores trigonometry through the context of fortification design.

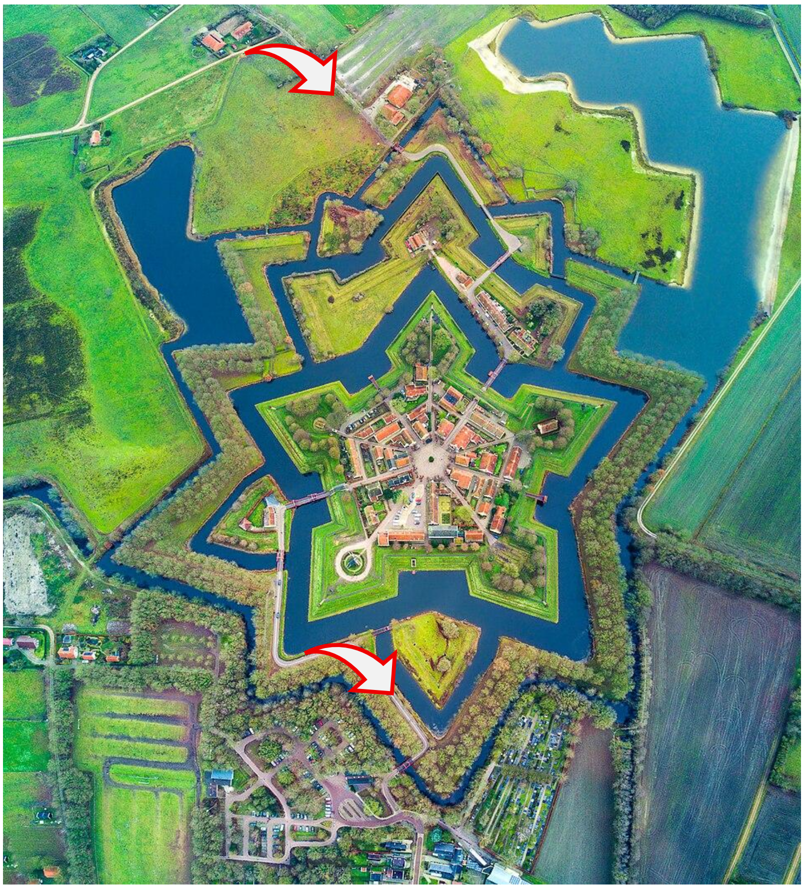

Figure 1 Aerial photo of Bourtange (credits: Dack9 - wikipedia, CC BY 4.0)

Introduction

Textbooks tend to assume that calculating the heights of church towers in the Netherlands relate closely to students' everyday lives, but in my experience, no student is actually interested in how tall that tower is. After many years of such calculations using sine and cosine in right-angled triangles, I felt ready for a more challenging application. I found this in the fortification engineering of the sixteenth and seventeenth centuries.
My own research showed that a wide range of calculation methods was in use by writers on fortification engineering. Authors with a mathematical background were especially skilled at varying dimensions, proportions, and formulas for angle calculation. They also explained, step by step, how to apply the sine rule and the tangent rule.

[1] An earlier version of this article has been published in the Dutch journal for mathematics teachers, Eulides 96-6. It is available online at https://www.nvvw.nl/wp-content/uploads/2024/03/096_2020-21_06_08-13_vestingbouw.pdf

The basic shape of a fortification with bastions is shown in Figure 2. In the seventeenth century, many books on fortification were published, ranging from beautifully illustrated picture books to engineering textbooks with an emphasis on calculations. They had a preference for regular polygons, from the square and the pentagon to the dodecagon and beyond, ignoring the need to adapt the fortification to the terrain conditions.

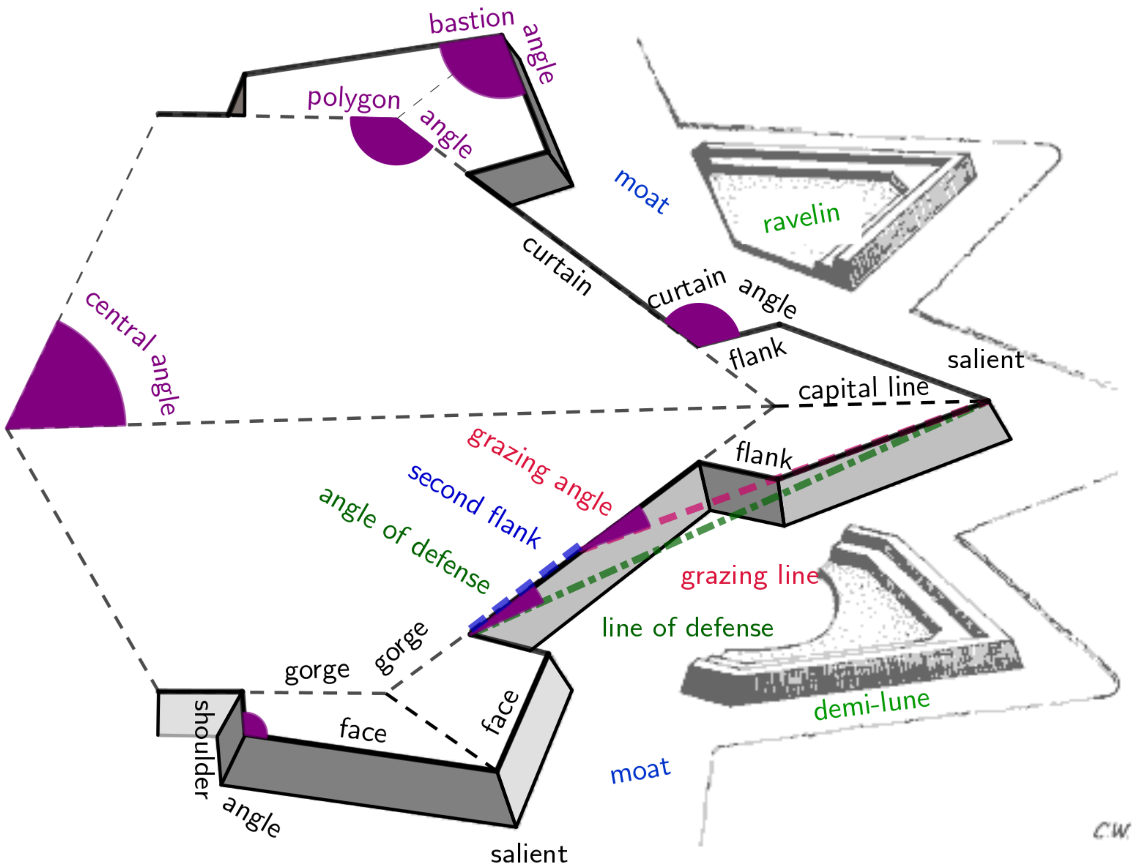

Figure 2 Kamps, van Kerkum, de Zee, *Terminologie verdedigingswerken*[2], 1999.

A *bastion* is the outward projection at the corners of a fortification. The tip of the bastion is called the *salient*. The wall between two bastions is called the *curtain*. The sides of a bastion are called *flanks*, and the front sides are called *faces*. The line between the bastion and the main fortification is called the *gorge*. Military engineers often draw this as a straight line, while mathematicians draw the gorge as an extension of the curtain. This highlights the side of the polygon and makes the calculations easier. The line from the salient to the gorge is called the *capital line*. The fortification is surrounded by a moat, in which ravelins lie like islands[1].

Two concepts are central to fortification design. In the seventeenth century, these were known as the *grazing line* and the *line of defense*. The line of defense runs from the corner point where the curtain meets the flank to the salient, and the grazing line runs along the face beginning at the intersection of the extended face with the curtain. The grazing angle is the acute angle between the curtain and the grazing line, while the defense angle is the acute angle between the curtain and the line of defense.

From a military point of view, it was crucial that these lines were no longer than the effective range of a musket: 60 rods[3], which is equivalent to 720 feet or 226 meters. That is why the lengths of these lines recur throughout the exercises in this document.

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[2] Translations from http://www.internationalfortresscouncil.org/mfd.html

[3] *Rod* is an historical unit of length. The Rhenish *rod* (Dutch: roede) was approximately 3,68 m. One rod was divided into 12 feet.

Simon Stevin

This article is motivated by the desire to commemorate Simon Stevin (1548–1620). In the seventeenth century the Dutch Republic became a laboratory for modern fortification. New ideas in mathematics and geometry were applied directly to military engineering. Prominent people such as Simon Stevin and Prince Maurits played a key role in turning theory into practice. The Eighty Years'War[4] began with religious unrest, dissatisfaction with administrative issues, persecution of heretics and taxation, and rejection of the Spanish king's authority over the Dutch provinces. In the end, the Dutch Republic became an independent country and a global maritime player. The Dutch Republic is the predecessor of the current Kingdom of the Netherlands.

Simon Stevin[5] (1548–1620) was a leading advocate of practical mathematics in the Dutch Republic. In *De Thiende* he promoted decimal fractions, and in his work on hydrostatics he argued that physical behavior follows mathematical laws. This approach was famously illustrated by his experiment at the New Church in Delft, where balls of different weights were shown to fall at the same rate.

Prince Maurits of Nassau[6] (1567–1625) was the head military commander remembered for reforming warfare. He insisted on discipline, training, and the systematic use of geometry in siege and fortification. Under his leadership, mathematical theory was translated into military practice, turning the Dutch army into one of the most modern forces of its time. Warfare changed from armies battling in the field to besieging towns and strongholds. Small fortresses appeared everywhere, protecting rivers, roads, canals. Figure 3 shows dozens of cities with their fortifications as well as dozens of small fortresses along the major rivers. They can be recognised by the small flags.

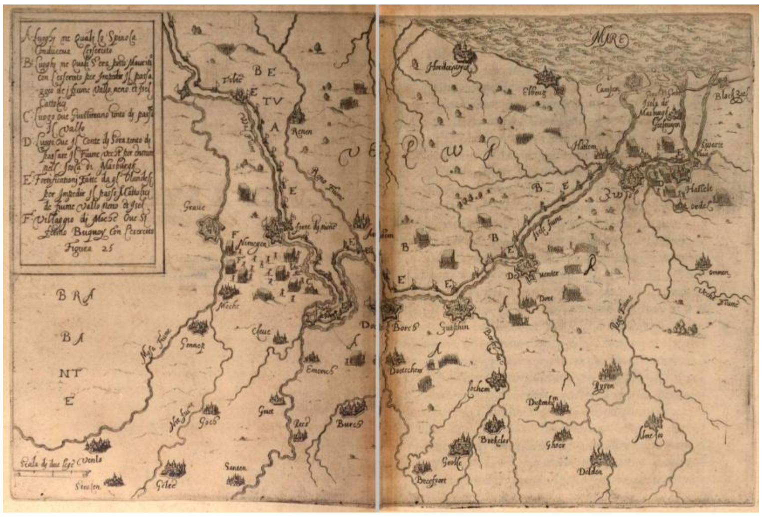

Figure 3 Pompeo Giustiniano, *Delle Gverre Di Fiandra Libri VI*[7].

Stevin was among the first to write a book in Dutch on fortification design, complete with all dimensions: lengths, heights, widths, and more. His design was based on the Italian system, featuring solid stone bastions over 10 meters high.

Casemates, multi-storey stone structures served as emplacements for cannons, were constructed at the corner of the flank and the curtain. . Positioned safely at the rear of the bastion, the casemates allowed defenders to fire along the walls towards the neighboring bastion, preventing an enemy from placing an explosive unnoticed near the ramparts or bastions.

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[4] Wikipedia offers a summary: https://en.wikipedia.org/wiki/Eighty_Years%27_War

[5].Wikipedia offers a short biography:: https://en.wikipedia.org/wiki/Simon_Stevin.

[6].Wikipedia offers a short biography: https://en.wikipedia.org/wiki/Maurice,_Prince_of_Orange

[7] Pompeo Giustiniano, Delle Gverre Di Fiandra Libri VI .... , Anverso, Trognesio, 1609, see : https://diglib.hab.de/drucke/sch-gf-2f-44/start.htm

Stevin's book *Sterctenbovwing*[8] is an excellent subject for a school research project. His detailed descriptions make it possible to create a paper model or a scale model of his design, as well as creating a 3D model for a 3D printer. However, this is too extensive for a quick assignment in a regular lesson.

Stevin is important to this story because he advised Prince Maurits on the curriculum of the *Duytsche Mathematique*. This program started in Leiden in 1600 in response to the need for well-trained military engineers. Fortification design was a serious educational goal. Stevin proposed that the course should use wooden scale models in which the grazing line could be made visible with strings. From 1610 to 1679, the teaching was carried out by members of the Van Schooten family. In addition, there were several private teachers in the Republic who taught mathematics and fortification design.

History

Clear examples of the Italian system can be found in the Utrecht bastions Zonnenburg and Manenburg, and in Maastricht in the round towers *Haet ende Nijt* and *De Vijf Koppen*. The Old Dutch fortification system was introduced by Stevin's contemporaries, such as Adriaan Anthonisz, mayor of Alkmaar and father of Adriaan Metius. They developed a system based on earthen ramparts, bastions only a few meters high, and cannons placed on the ramparts and bastions rather than in casemates, all surrounded by wide, water-filled moats. Compared to the heavy stone construction of the Italian system, such fortifications could be realized much more quickly and at a far lower cost. The scale of the difference becomes striking when Stevin's dimensions are converted from feet to meters: a bastion 40 feet high exceeds 12 meters, and a moat 55 feet deep is more than 17 meters deep. See Figure 4 for Stevin's sketch.

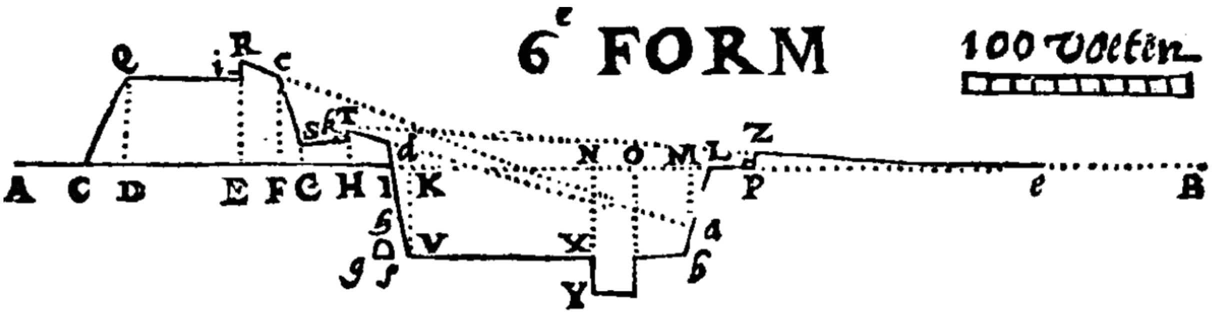

Figure 4 Simon Stevin, *Sterctenbovwing*[9], 1594.

This opens up an interesting question: where in the Netherlands or Belgium could such a structure realistically be built? At the end of the seventeenth century, Menno van Coehoorn introduced the New Dutch fortification system, with further refinements to the design of flanks, ravelins, and other features within the moat.

The complete history of fortification is a subject in its own right, but it can be captured in a few sentences. In the sixteenth century, the French kings and Charles V (Holy Roman Emperor, King of Spain, and ruler of the Low Countries) fought over the Italian city-states. These cities were defended by walls, bastions, and often dry moats. Usually the besiegers were victorious, but in some cases the defenders held out. From this practice, the Italian fortification system emerged.
During the Eighty Years' War, Dutch cities had neither the time nor the resources to construct stone bastions. Earthen ramparts could be built much more quickly and cheaply, and they turned out to be remarkably resistant to bombardment. A more detailed discussion is available on my website[10].

[8] *Sterctenbovwing* is available online at https://mdz-nbn-resolving.de/urn:nbn:de:bvb:12-bsb00103429-8
[9] There is an animation on my website that shows all the dimensions.
https://www.fransvanschooten.nl/StevinGeoGebra.htm#verheventeyckening
[10] See http://www.fransvanschooten.nl/Stevin.htm

In this article, I focus on a single example of the Italian system, because it allows for clear and meaningful calculations. From a geometrical perspective, the Old Dutch fortification system is very appealing. It appears simple, yet turns out to be more challenging than expected. A number of exercises have been selected, all based on a regular pentagon. For each exercise, I give the name of the author and the title of the book, which are the historical sources that I used.

Alghisi

The first exercise is taken from Galasso Alghisi da Carpi (1523–1573), an Italian Renaissance architect. He believed that fortification should not be left to military men alone and wrote a book describing how a fortress with bastions ought to be designed. In *Delle Fortificationi*[11] he discusses polygons ranging from the pentagon to the 21-gon. This is mathematics and urban design, far removed from military practice.

The pentagonal fortress is laid out along the lines of a regular decagon. From this drawing, questions can be asked about the sizes of relevant angles, such as the bastion angle *DAH* or the angle *AKC* at the bend in the curtain. For this exercise, it is sufficient to specify the area of the inner pentagon *KLMNP*, for instance one morgen[12], which equals exactly 600 square rods[13]. The task is then to calculate the length of the rampart *KR* and the sides of the face of a bastion *AQ*; see Figure 5.

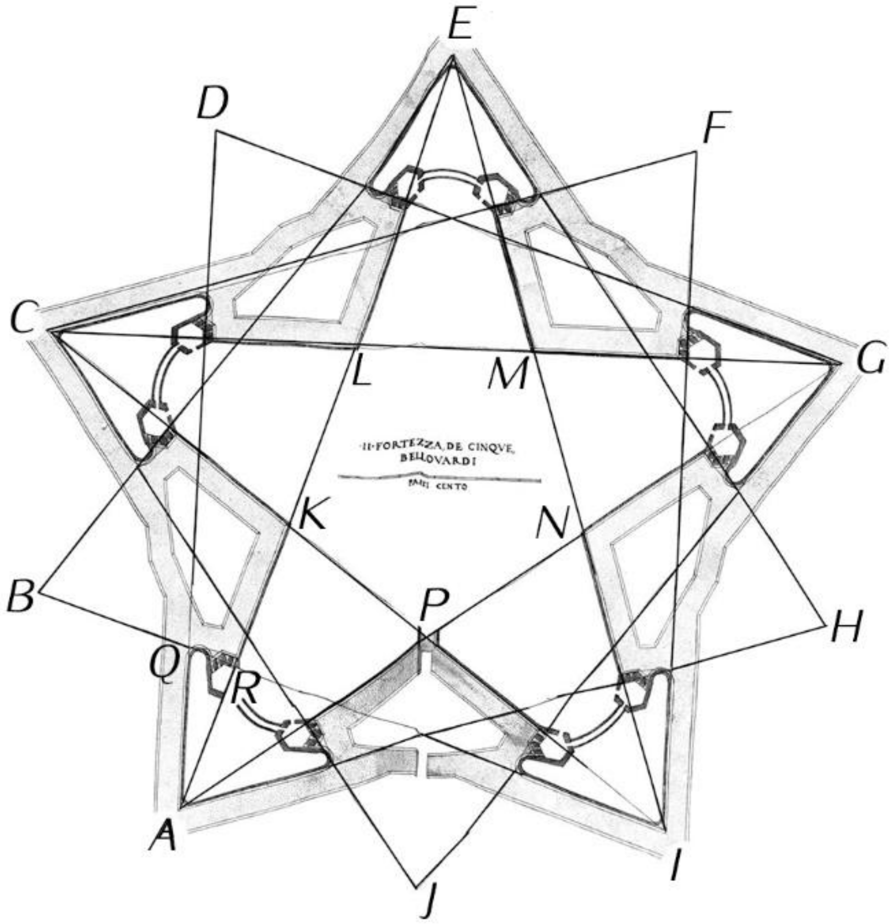

Figure 5 Galasso Alghisi da Carpi, *Delle fortificationi*, 1570

[11] *Delle Fortificationi* is available online at https://www.e-rara.ch/zut/content/structure/234496

[12] The *Morgen* is an historical unit of area. The Rhenish *morgen* was a Dutch unit of area equivalent to 8500 m^2.

[13] *Rod* is an historical unit of length. The Rhenish *rod* (Dutch: roede) was approximately 3,68 m.
Therefore one square *roeden* is about 14,2 m^2 and one *morgen* is 600 square *roeden*. One square *roeden* is 144 square *feet*.

Goldmann

Nikolaus Goldmann (1611–1665) gave private lessons in Leiden during the period when the Van Schooten family taught at the *Duytsche Mathematique*. Using his drawing from *La nouvelle fortification*[14], students can, for instance, be asked to construct it to scale, with the curtain wall *ST* equal to 40 rods, the flank *RS* perpendicular to the curtain and 6,7 rods long, the gorges *aS* and *Tb* each 9,2 rods long, and the grazing line *AM* intersecting the curtain at its midpoint; see Figure 6.

A follow-up task could involve calculating the size of the bastion angle *PAR*, and the length of the distance *AX* between two neighboring bastions, or the area of a bastion. The construction may be done on paper or with GeoGebra[15]. A key advantage of GeoGebra is that students arrive at the answers while drawing. When a polygon is constructed in GeoGebra, an approximate value for its area is given automatically. Students must still formulate the calculation themselves, of course, but it helps to know the target result. Drawing also encourages students to explore auxiliary lines and planes. Goldmann included several clever auxiliary lines in his drawing, which means the problem can be solved without using the sine rule.

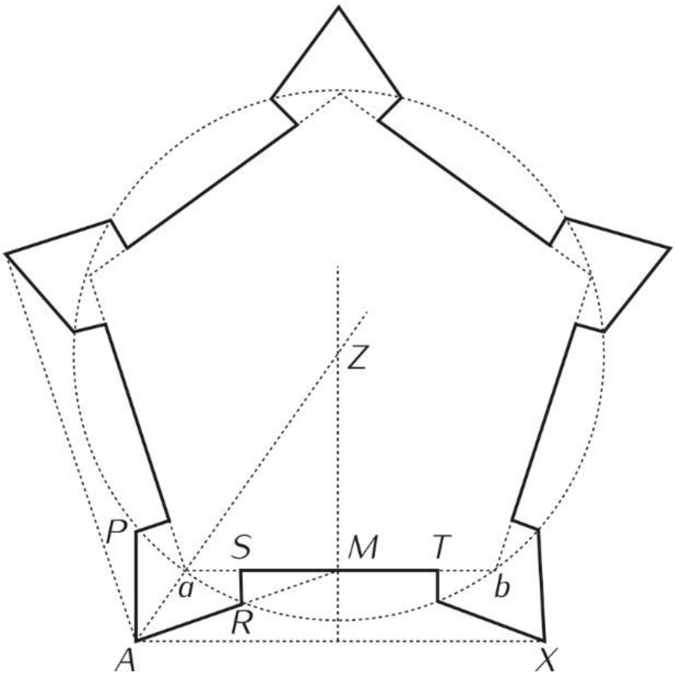

Figure 6 after Nikolaus Goldmann, *La nouvelle fortification*, 1643

Metius

Adriaan Metius (1571–1635) studied mathematics in Franeker and in Leiden. His father, Adriaen Anthonisz (1541–1620), was a fortification engineer, cartographer, mathematician, and mayor of Alkmaar. Adriaen Anthonisz worked according to the principles of the Old Dutch fortification system and was active throughout the northern Netherlands. In his book *Fortificatie ofte stercken-bouwinghe*[16] he incorporated many of his father's ideas.

This exercise also deals with a regular pentagon; see Figure 7. Given a fortress with five bastions and side *ON* equal to 540 feet[17]. The proportions are: gorge : side = 2 : 9 and flank : side = 1 : 6. The task is to calculate the

[14] *La Nouvelle Fortification* is available online at https://doi.org/10.3931/e-rara-46786

[15] GeoGebra is a free interactive geometry, algebra, statistics and calculus application, intended for learning and teaching mathematics and science from primary school to university level. See https://www.geogebra.org/

[16] *Fortificatie ofte stercken-bouwinghe* is available online at https://books.google.nl/books?id=xKLDSKQ0CVgC

[17] Twelve feet in one rod, one feet is 0,307 meters.

sizes of the angles, for example the bastion angle at *H*, and the lengths of the various elements, such as the length of face *DH*, and the lengths of the grazing line *HQ* and the line of defense *AH*. This exercise can also be solved without using the sine rule when auxiliary triangles are added. Follow-up tasks could involve working out a hexagonal or heptagonal fortress using the same assumptions. Metius continued this approach up to the dodecagon.

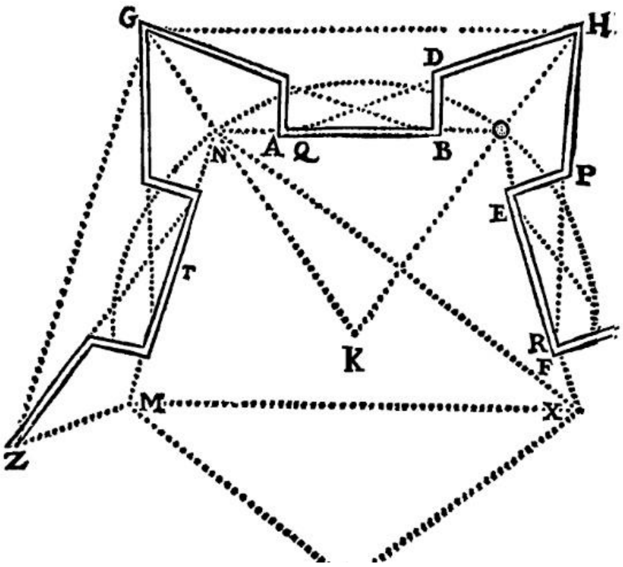

Figure 7 Adriaan Metius, *Fortificatie ofte stercken-bouwinghe*, 1626

Pieter van Schooten

Pieter van Schooten was the third member of the Van Schooten family to teach at the *Duytsche Mathematique*, serving from 1660 until 1679. The drawing in Figure 8 is taken from the Leiden manuscript BPL 1993[18].

Pieter constructed a regular pentagonal fortress with a bastion spacing *CI* of 60 rods. He determined the size of the bastion angle *FIY* as one third of the central angle *ABZ* plus 30°. The length of the capital line *AC* is in the ratio 1 : 3 to side *AB*, while the length of the gorge *AO* is in the ratio 1 : 5 to side *AB*.

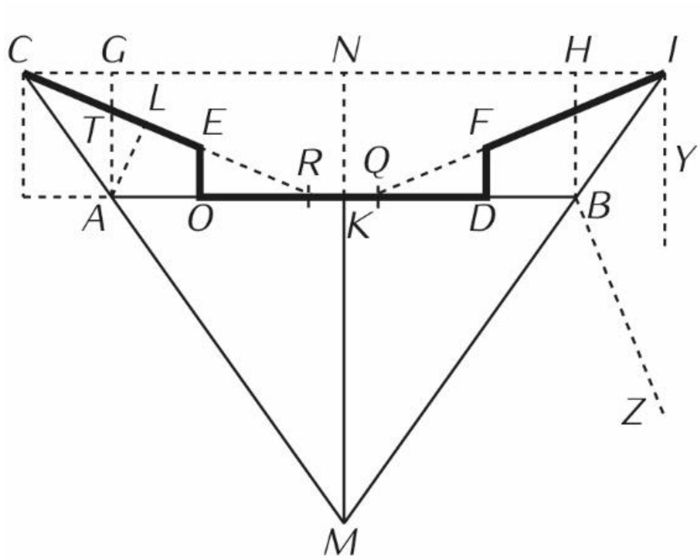

Figure 8 after: Petrus van Schooten, *BPL 1993*, *Versterkingskunst (belgice).*

[18] BPL 1993 is available online at https://digitalcollections.universiteitleiden.nl/view/item/876666

A possible task is to calculate the size of the angle of defense *ADC* and the length of the grazing line *CR*.

Mathematically, this is an elegant problem because, after the initial exercise of calculating the various angles, one has to formulate an equation for the distance between the bastion points. If we denote the unknown length of the capital line *AC* by x, then the length of side AB is equal to $3x$.
This leads to the equation $x \sin\ 36° + x + x \sin\ 36° = 60$.
An alternative strategy is to first calculate the angles and the various lengths of a fortress which has a side length 10, and then to determine the correct scaling factor. Using this strategy, one can get quite far!

Cellarius

Andreas Cellarius became rector of the Latin school in Hoorn in 1637. He died in 1665, five years after the publication of his magnificent *Harmonia Macrocosmica*[19] devoted to the motion of the heavens.

Cellarius constructed in his *Architectura militaris*[20] a regular pentagon with a curtain *CG* of 36 rods, a flank *CI* of 9 rods, and a face *MI* of 24 rods. The bastion angle *IMZ* was calculated as two thirds of the central angle *CBX*, but was restricted to a maximum of 90°; see Figure 9.

Cellarius demonstrated his mathematical expertise by applying the tangent rule. This rule forms a valuable addition to the sine rule and the cosine rule. When the angle between two known sides is given, the cosine rule can be used to calculate the length of the unknown opposite side, while the tangent rule allows the two remaining unknown angles to be determined. With the tangent rule, these angles can be found in one step, instead of using the cosine rule to compute the third side and then applying the sine rule to find the remaining angles.

Tangent rule: In any triangle *ABC* with angles α, β en γ and sides *AB* = *c* , BC = *a* and AC = *b*,

$$\frac{a-b}{a+b} = \frac{\tan\left(\frac{1}{2}(\alpha-\beta)\right)}{\tan\left(\frac{1}{2}(\alpha+\beta)\right)}.$$

When the lengths of two sides a and b and the size of the included angle $\gamma$are known, the sum of the remaining two angles can be found from the fact that the angles of a triangle add up to 180°: $\alpha + \beta = 180° - \gamma$. The tangent rule can then be used to determine the unknown value $\frac{1}{2}(\alpha - \beta)$. Since $\alpha = \frac{1}{2}(\alpha + \beta) + \frac{1}{2}(\alpha - \beta)$ and $\beta = \frac{1}{2}(\alpha + \beta) - \frac{1}{2}(\alpha - \beta)$, the sizes of the unknown angles α and β follow directly.

The task associated with Figure 9 is to calculate the size of the angle of defense. One possible approach is to first determine the sizes of all angles, then to calculate the lengths of the various sides, including the capital line *BM* and the gorge *BC*, and finally to use the tangent rule in triangle *BMG* to find the angle of defense.

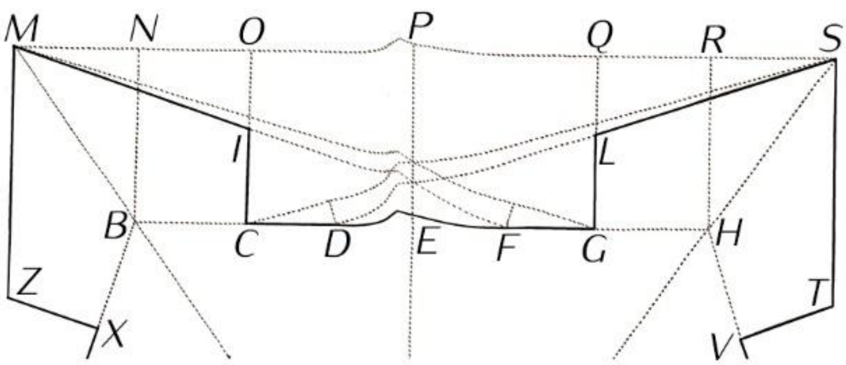

Figure 9 after: Cellarius, *Architectura militaris*, 1645

[19] *Harmonia Macrocosmica* is available online at http://www.flandrica.be/flandrica/items/show/1036
[20] *Architectura militaris* is available online at https://reader.digitale-sammlungen.de/resolve/display/bsb10863815.html

Comparison

The table below compares the structure of the exercises discussed. In this article I discuss only four examples, but in my research with Charles van den Heuvel[21] I studied the approaches of fifteen authors and 25 books or manuscripts. These are filled with trigonometric calculations. In the exercises presented here, two different formulas are used for the bastion angle, but my survey distinguishes more formulas:

- the bastion angle is obtuse (as Stevin)
- the bastion angle is a right angle
- the bastion angle is 15° more than half of the central angle (as Goldman)
- the bastion angle is 20° more than half of the central angle
- the bastion angle is 30° more than one-thirds of the central angle (as Pieter van Schooten)
- the bastion angle is two-thirds of the central angle (as Metius and Cellarius)

The following table shows the input data with which the four authors begin their designs or exercises. It is striking how much variation mathematicians managed to introduce to calculate the dimensions of similar fortresses.

	Nikolaus Goldmann	Adriaan Metius	Pieter van Schooten	Andreas Cellarius
Bastion angle as a fraction of the central angle		two-thirds	one-thirds + 30°	two-thirds
Side		540 foot	capital : side = $\frac{1}{3}$	
Distance between two saliants			60 rods	
Curtain	40 rods			36 rods
Flank	6,7 rods	flank : side = $\frac{1}{6}$		9 rods
Gorge	9,2 rods	gorge : side = $\frac{2}{11}$	gorge : side = $\frac{1}{5}$	
Face				24 rods
Capital Line			capital : side = $\frac{1}{5}$	
Grazing Line	mid of curtain			
Line of Defense				

Conclusion

In the seventeenth century, mathematicians became deeply involved in fortification design, and military engineers were expected to master trigonometric calculations. Teachers with a solid mathematical background developed their own methods and formulas. Some of these books were reprinted for many decades and translated into several languages. The exercises discussed are challenging, but they can be used at the end of a course as a mathematical thinking exercise. The historical context helps to highlight the relevance of the sine rule and the cosine rule. Exercises and worked solutions are available on my website[22].

About the author

Henk Hietbrink enjoyed many years as a mathematics teacher. For the time being, however, he has stepped away from teaching to focus on muqarnas and sundials.

[21] Comparison of calculation schemes: https://www.fransvanschooten.nl/TechNet.htm
[22] Seven exercises are available at: https://www.fransvanschooten.nl/StevinWorkshop.htm